Seismometry Overview

This section presents an examination of the processes involved in the translation from ground motion into seismograms — removing the "black box." Shearer's sub-chapters 11.1 and 11.3 parallel this discussion. A chapter in the mechanics book by Symon presents the basis for the math behind the solution of the harmonic oscillator equation, which is our model for the seismograph. Aki & Richards, chapter 10, covers the same material along with the application to a seismograph.

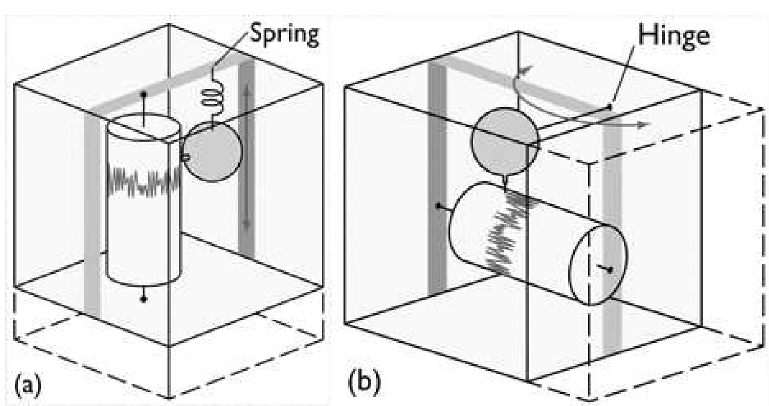
Seismograph

A seismograph produces a permanent recording of ground motion. Because ground motion is a vector, one needs three independent components to get a complete recording. One usually pictures these as a vertical component and two orthogonal horizontal components, but they only need to be independent. To cut costs and to simplify processing, many networks of seismograph stations use only single-component (usually vertical) seismographs.

A seismograph has three stages (convolution)

- **Seismometer**: here modeled as a harmonic oscillator that reacts in a predictable way to ground motion
- **Sensor**: converts the mechanical output from the seismometer into a form of energy that can be recorded
- **Recorder**: produces a permanent time history of the ground motion

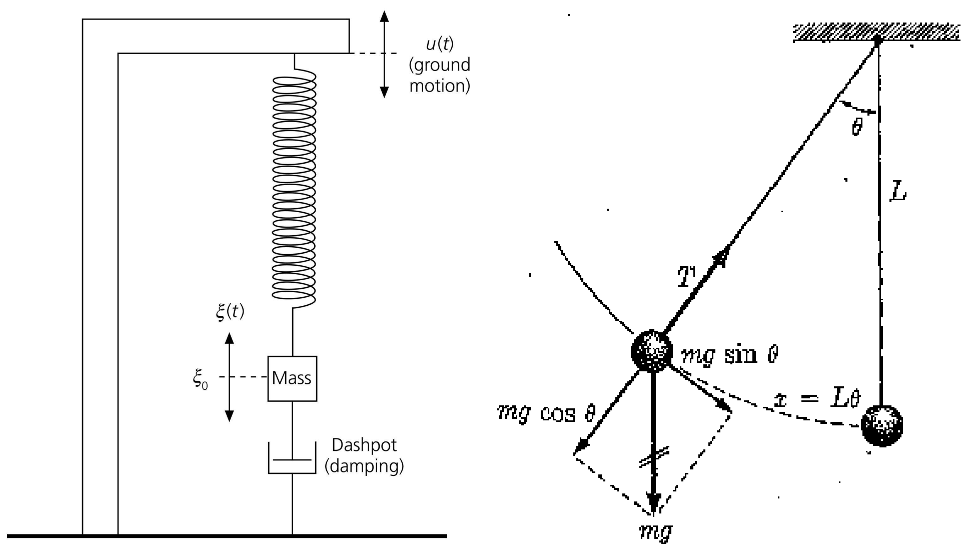
Simplified vertical and horizontal seismographs



Seismometer, sensor, and recording are all included. **(a)** vertical-motion seismograph. **(b)** 20.8)

A seismometer is a damped, driven linear oscillator

Seismometers are designed so that their motion $x(t)$ is constrained to be in a single direction. The general solution for that motion is of the form $x(t) = C_1 x_1(t) + C_2 x_2(t) + x_P(t)$, where x_1 and x_2 are solutions of the homogeneous equation, x_P a particular solution of the inhomogeneous equation, and C_1 and C_2 are arbitrary constants which can be determined given two "boundary" conditions. A *homogeneous* equation is one for which there is no driving term.



Shown in the left panel above is a damped spring, which is the prototype for the vertical seismometer. Leaving out the damping and driving terms, Newton's second law then gives

$$m\ddot{x} = -kx \Rightarrow \ddot{x} + \omega_0^2 x = 0,$$

where k is the spring constant and $\omega_0^2 = k/m$.

Shown in the right panel in the figure above is a pendulum, for which gravity provides the restoring force — the prototype of a horizontal seismometer.

For the pendulum, the motion is in the $\hat{\theta}$ direction (positive $\hat{\theta}$ is clockwise for the above figure). Newton's second law for this problem is

$$I\ddot{\theta} = \text{Total torque} = -mgL\sin\theta,$$

where I is the moment of inertia, which for a "perfect" pendulum is given by $I = mL^2$. For small deflections, $\sin\theta \approx \theta$, and $\hat{\theta}$ is approximately along a horizontal direction. Newton's second law for the pendulum is then

$$mL^2\ddot{\theta} + mgL\theta = 0 \Rightarrow \ddot{\theta} + \omega_0^2\theta = 0,$$

where $\omega_0^2 = g/L$.

Hence, the motion of both vertical and horizontal (undriven and undamped) seismometers obey the same equation. To complete the picture, we need to add the damping and driving terms.

- Damping: Friction opposes motion, dissipates energy. Assume linear in velocity $\Rightarrow$ of form $-b\dot{x}$
- Driving Force of form $F(t)$.
- $\Rightarrow m\ddot{x} + b\dot{x} + kx = F(t)$, an inhomogeneous, second-order, linear ordinary differential equation with constant coefficients.

The final equation is that of a driven, damped linear harmonic oscillator. I am told that this is covered in MATH 2214, so what I give below is (hopefully) review. I start by deriving the solutions to the homogeneous equation. Then we bring in the driving term.

Solutions to the homogeneous equation

$$\ddot{x} + 2\gamma\dot{x} + {\omega_0}^2 x = 0, \tag{1}$$

where ${\omega_0}^2 = \frac{k}{m}$, and $\gamma = \frac{b}{2m}$

Assume solution of form $x = H(t)\,e^{pt}$, where $H(t)$ is the Heaviside or step function introduced earlier. Substituting this into equation (1) leads to a quadratic equation for p

$$p^2 + 2\gamma p + {\omega_0}^2 = 0,$$

which has solutions

$$p = -\gamma \pm \sqrt{\gamma^2 - \omega_0^2}.$$

This then leads to

$$x(t) = H(t)\,e^{-\gamma t \pm t\sqrt{\gamma^2 - {\omega_0}^2}}$$

Case 1: Underdamped

$${\omega_0}^2 > \gamma^2$$

Then, for $t > 0$,

$$x(t) = C_1 e^{-\gamma t + i\omega_1 t} + C_2 e^{-\gamma t - i\omega_1 t},$$

where $\omega_1 = \sqrt{{\omega_0}^2 - \gamma^2}$. x must be real, so

$$x = Ae^{-\gamma t}\cos(\omega_1 t + \theta),$$

with $C_1 = \frac{1}{2}Ae^{i\theta}$, and $C_2 = \frac{1}{2}Ae^{-i\theta}$, which is a damped, oscillating function.

Case 2: Overdamped

$${\omega_0}^2 < \gamma^2$$

Then, for $t > 0$,

$$x(t) = C_1 e^{-\gamma_1 t} + C_2 e^{-\gamma_2 t},$$

where $\gamma_1 = \gamma + \sqrt{\gamma^2 - {\omega_0}^2}$ and $\gamma_2 = \gamma - \sqrt{\gamma^2 - {\omega_0}^2}$. There is no oscillation. Because $\gamma_1 > \gamma > \gamma_2$, the first term falls faster than the second.

Case 3: Critically Damped

$${\omega_0}^2 = \gamma^2$$

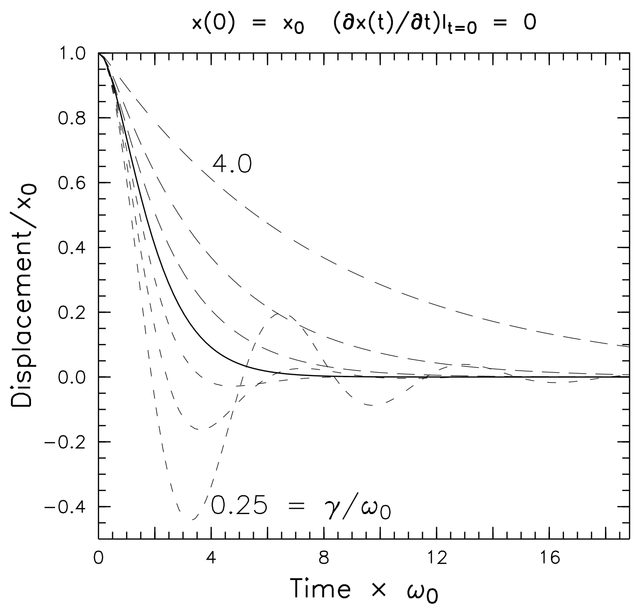
Then, for $t > 0$,

$$x(t) = C_1 e^{-\gamma t} + C_2 t e^{-\gamma t}.$$

(Verify that the second term is a solution.)

Solving the homogeneous equation: an example (Symon: 2–25)

A mass m subject to a linear restoring force $-kx$ and damping $-b\dot{x}$ is displaced by a distance x_0 from equilibrium and released with zero initial velocity. Find the analytical solutions and plot the motion in the underdamped, critically damped, and overdamped cases.



For the initial conditions of $x(0) = x_0$ and $\dot{x}(0) = 0$, the solutions for the three cases are

- **underdamped**: $x(t) = H(t)x_0 e^{-\gamma t}\left[\cos\omega_1 t + \frac{\gamma}{\omega_1}\sin\omega_1 t\right]$ (The easiest way to get this answer is to work with $H(t)Ae^{-\gamma t}\cos(\omega_1 t + \theta) = H(t)e^{-\gamma t}[C\cos\omega_1 t + D\sin\omega_1 t]$, which was gotten by using the trig identity $\cos(a+b) = \cos a\cos b - \sin a\sin b$, and setting $C = A\cos\theta$ and $D = -A\sin\theta$.)
- **critically damped**: $x(t) = H(t)x_0 e^{-\omega_0 t}[1+\omega_0 t]$
- **overdamped**: $x(t) = H(t)\frac{x_0}{1-\frac{\gamma_1}{\gamma_2}}\left[e^{-\gamma_1 t} - \frac{\gamma_1}{\gamma_2}e^{-\gamma_2 t}\right]$

All these solutions should be multiplied by the Heaviside function, as the solutions are not valid for negative time.

Note that because $\dot{x}(0) = 0$, x_0 is an extremum. Also, only for the underdamped case is $x(t)$ ever negative. Except for $\gamma = 0$, $x(t) \to 0$ as $t \to \infty$. Note also that the quantities plotted are dimensionless: the amplitude is then in units of x_0 and the time is scaled by the natural period.

Solving the inhomogeneous equation

$$\ddot{x}+2\gamma\dot{x}+{\omega_0}^2x=\frac{F(t)}{m},$$

We define $\frac{F}{m}=-\ddot{u}_x$, where u_x is the displacement (ground motion) of the system in the $\hat{x}$ direction. Thus

$$\ddot{x}+2\gamma\dot{x}+{\omega_0}^2x=-\ddot{u}_x.$$

Using Fourier analysis, any $u_x(t)$ can be written as the real part of a sum over terms of the form $U_0(\omega)e^{-i\omega t}$ with $U_0(\omega)=|U_0|\,e^{-i\theta}$.

For this driving term, we assume $x(t)=X(\omega)e^{-i\omega t}=\xi(\omega)u_x(t)$.
The equation for $\xi(\omega)$ is then

$$-\omega^2\xi-2i\gamma\omega\xi+{\omega_0}^2\xi=\omega^2.$$

Solution

$$\xi(\omega)=\frac{\omega^2}{{\omega_0}^2-\omega^2-2i\gamma\omega}$$

Limiting Cases:

- Rapidly oscillating u_x: $\omega >> \omega_0$, $\xi(\omega)\Rightarrow -1$. So $x(t)\Rightarrow -u_x(t)$. In other words, $\ddot{x}$ term dominates and x exactly out of phase with u_x
- Slowly oscillating u_x: $\omega << \omega_0$, $\xi(\omega)\Rightarrow \frac{\omega^2}{{\omega_0}^2}$, leading to $x(t)\Rightarrow -\frac{\ddot{u}_x}{{\omega_0}^2}$.

Amplitude and Phase

$$\xi(\omega)=\frac{\omega^2}{{\omega_0}^2-\omega^2-2i\gamma\omega}=|\xi(\omega)|\,e^{i\phi(\omega)}$$

Amplitude:

$$\sqrt{\xi(\omega)\xi^*(\omega)}\equiv|\xi(\omega)|\,,\ \text{so}\ |\xi(\omega)|=\frac{\omega^2}{\sqrt{({\omega_0}^2-\omega^2)^2+4\gamma^2\omega^2}}$$

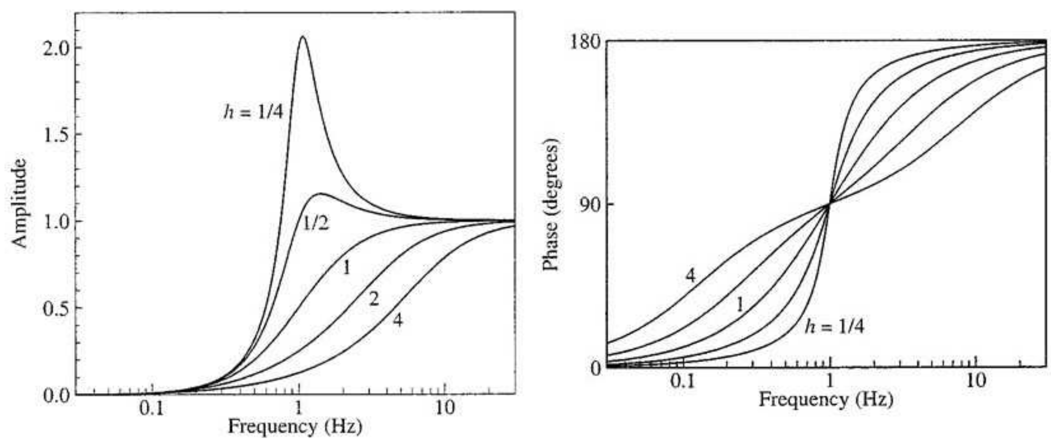
Phase:

$$\phi(\omega) = \tan^{-1}\left[\frac{\mathrm{Im}\left[\xi(\omega)\right]}{\mathrm{Re}\left[\xi(\omega)\right]}\right]$$

Using

$$C = \frac{1}{a+ib} = \frac{a-ib}{a^2+b^2},$$

$$\frac{\mathrm{Im}C}{\mathrm{Re}C} = \frac{-b}{a}, \;\; \phi(\omega) = \tan^{-1}\left[\frac{2\gamma\omega}{{\omega_0}^2-\omega^2}\right]$$



The amplitude (left) and phase (right) response functions for a seismometer of 1 Hz natural resonance ($\omega_0 = 2\pi$) at various levels of damping, with $h = \gamma/\omega_0$. If $\omega_0 = 2\pi$, $h = 1$ for critical damping. (Shearer, Fig. 11.2)

Defining $\omega = 2\pi\nu$, the above equations for amplitude and phase can be written as

$$|\xi(2\pi\nu)| = \frac{\nu^2}{\sqrt{(1-\nu^2)^2+4h^2\nu^2}} \;\text{ and }\; \phi(2\pi\nu) = \tan^{-1}\left[\frac{2h\nu}{1-\nu^2}\right].$$

For zero damping ($h = 0$), the amplitude at resonance ($\omega = \omega_0$) is infinite and the phase is zero for all frequencies below 1 Hz and 180° for all frequencies above 1 Hz. When troops are crossing a small bridge, they often break step because if their marching cadence were near the natural frequency of the bridge, in-step marching could start the bridge oscillating.

On the next page are three figures from *Earthquake Rose*. Shown is a sand-tracing pendulum, located at a shop in Port Townsend, and the image drawn by that pendulum during the February 28, 2001 M=6.8 earthquake beneath nearby Olympia. The bottom figure has been digitally enhanced to make it easier to see details.

Solving the inhomogeneous equation: Symon: 2.37(a)

This example shows how one can find a solution for a driven harmonic oscillator for which the driving force is given. If one plotted the solution as a function of time (only for t $>$ 0), one would see at first a “transient” solution which is governed by the initial conditions, followed later by a “steady-state” solution which is governed by the driving force.

> Find, using the principle of superposition, the motion of an underdamped oscillator, with $\gamma = \frac{1}{3}\omega_0$, initially at rest and subject, after time $t = 0$ to a force $F(t) = A\sin\omega_0 t + B\sin 3\omega_0 t$. ω_0 is the natural frequency of the oscillator.

$$m\ddot{x} + b\dot{x} + kx = H(t)\left[A\sin\omega_0 t + B\sin 3\omega_0 t\right],$$

where $\omega_0 = \sqrt{\frac{k}{m}}$, $\gamma = \frac{1}{3}\omega_0$, and $\omega_1 = \sqrt{\omega_0^2 - \gamma^2} = \frac{2}{3}\sqrt{2}\omega_0$. The solution is gotten by starting from Symon’s Eq (2–165) which is the complete solution for a sinusoidal applied force (his Eq. (2–149)).

$$x(t) = H(t)\left[Ce^{-\gamma t}\cos(\omega_1 t + \theta) + G_1\cos(\omega_0 t + \beta_1) + G_2\cos(3\omega_0 t + \beta_2)\right],$$

where $G_1 = -\frac{A/m}{\frac{2}{3}\omega_0^2}$, $G_2 = -\frac{B/m}{2\sqrt{17}\omega_0^2}$, $\beta_1 = 0$, and $\beta_2 = \tan^{-1}[-4]$.

Note that C and θ are the only unknowns. Now,

$$0 = x(0) = C\cos\theta + G_1 + G_2\cos\beta_2 \equiv C\cos\theta + H,$$

and

$$0 = \dot{x}(0) = -\gamma C\cos\theta - \omega_1 C\sin\theta - 3\omega_0 G_2\sin\beta_2 \equiv -\gamma C\cos\theta - \omega_1 C\sin\theta + J.$$

Hence

$$C\cos\theta = -H,$$

and

$$C\sin\theta = \frac{-J + \gamma H}{\omega_1} \equiv L.$$

So,

$$C = \sqrt{C^2\left[\cos^2\theta + \sin^2\theta\right]} = \sqrt{H^2 + L^2},$$

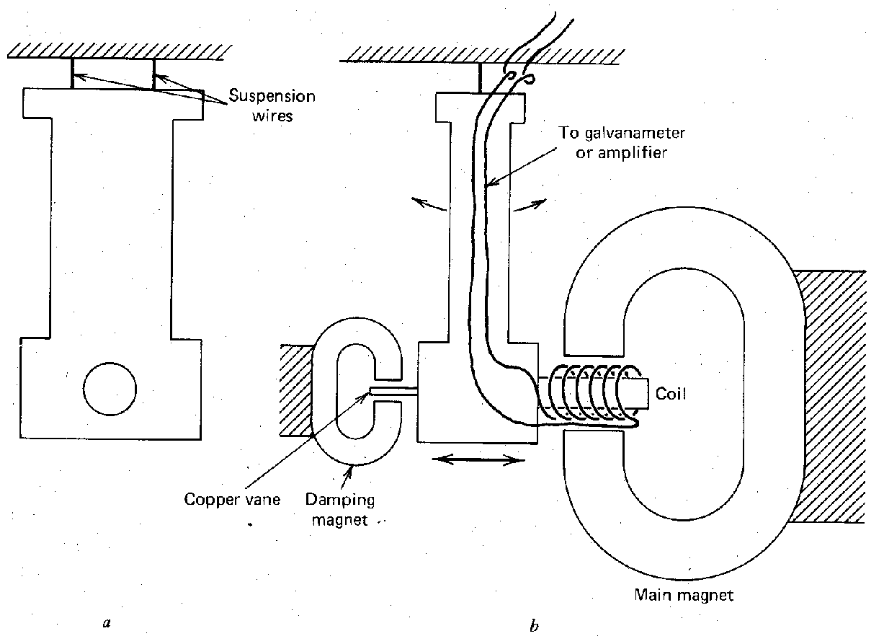
and

$$\theta = \tan^{-1}\left[\frac{L}{-H}\right].$$

Sensors: Velocity Transducer

Wrap (coil) wires around moving part of the seismometer, and construct a magnet so coil goes through it. By Faraday's Law of induction, an *EMF* is generated causing a current to flow through the coil. The *EMF*, and hence the current, is linear in $\dot{x}$.

If $x(t) \sim e^{-i\omega t}$, then $\dot{x} \propto -i\omega x$



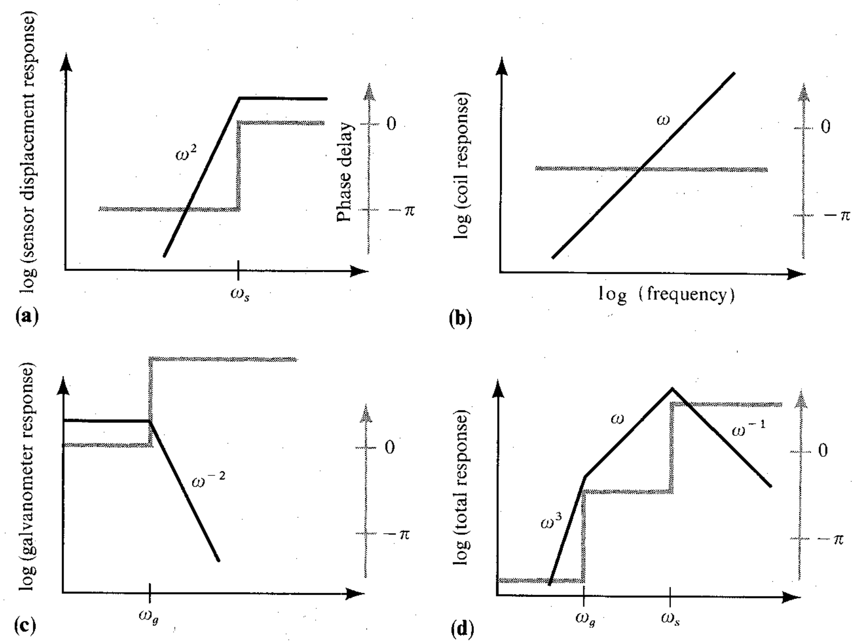
Principle of the horizontal "swinging gate" seismograph. *(a)* front view, *(b)* side view showing the sensor (DeBremacher, Fig. 7–4).

Recording

- **Mechanical**: Current acts as a driving term for a system satisfying the same equation of motion as the seismometer except no ω^2 on right-hand side. Acts as a low-pass filter. Example: galvanometer.

- **Digital**: In principle independent of frequency. In practice must guard against aliasing so must apply a low-pass filter before digitizing.

Integration

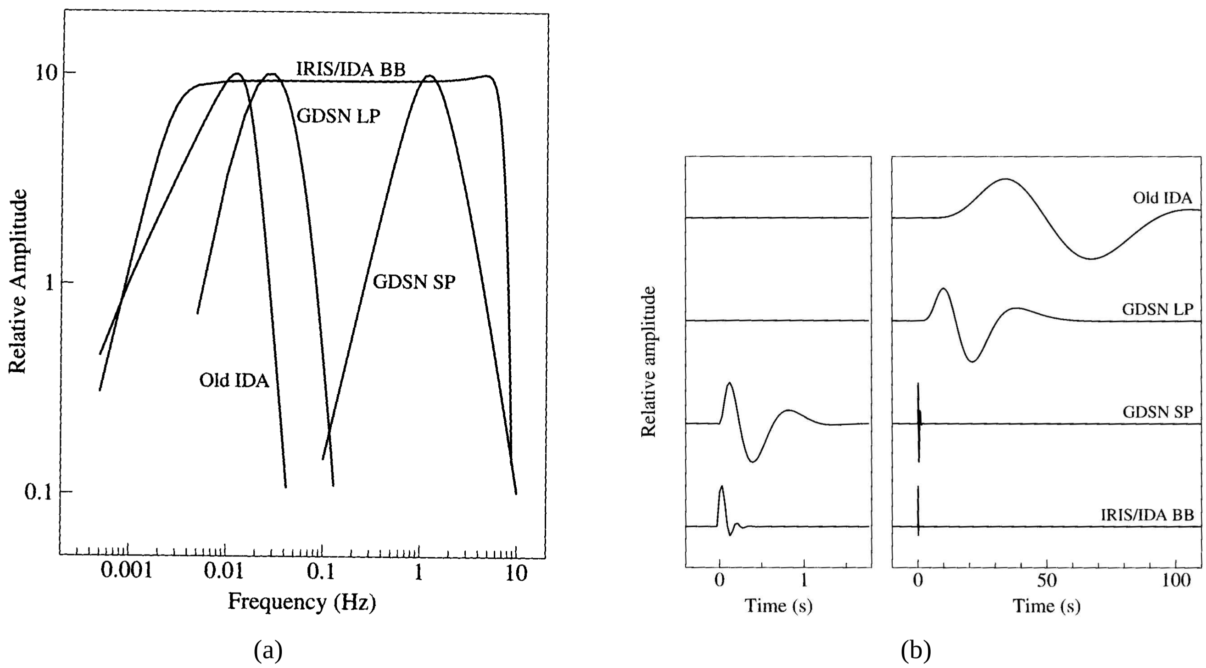


If negligible feedback between components, can just multiply "stages" together in the frequency domain. **(a)** simplified seismometer's displacement response. Amplitude is thick line, phase is thin line. **(b)** (coil) sensor response. **(c)** galvanometer or pre-digitizing low-pass filter response. **(d)** composite response obtained by multiplying the amplitudes and summing the phases. (Aki & Richards figure on p. 517.)

Multiplication in the frequency domain $\Leftrightarrow$ convolution in the time domain.

The net effect is an inverted U response curve for a traditional seismograph. Current models use sophisticated feedback mechanisms in their design to extend the response over a broader frequency band. Typical response for a modern broadband seismograph is to be flat in velocity response from 50 s period to over 10 Hz.

Effects of instrument response on waveforms



(a): Velocity response for four different seismographs; **(b)**: Impulse response for those seismographs. Input is a delta-function spike. (from Shearer: Figs. 11.4 and 11.5)

The Fourier transform of a spike in the time domain is a constant amplitude in the frequency domain. The IRIS/IDA BB response approximates this over the frequency band of normal interest. The instrument response is a causal filter, so for its output no "signal" arrives before the infinite-frequency calculated arrival time. With deviations from a constant amplitude response comes phase shifts. These both distort the arrival and shift the apparent arrival time to later times. Picking arrival times from Old IDA seismograms would give systematically late arrivals.

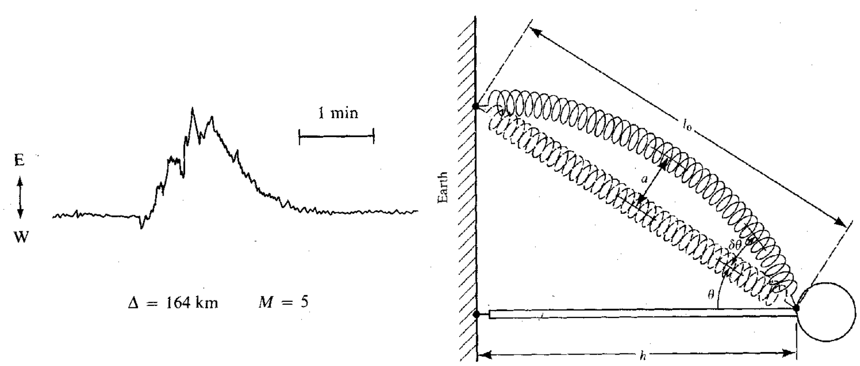
Dynamic Range

To capture faithfully ground motion from all earthquakes of interest requires recording displacements from milimicrons to millimeters at all frequencies of interest. (Recall magnitude $\propto \log_{10}$ of amplitude.)

For all seismographs there is only a limited dynamic range of motion over which the response will be linear with ground motion.

Improved low-frequency response requires longer pendulums, larger springs. Tricks like "zero-length" springs used to make more compact. For vertical seismographs, having spring at an angle increases free period. New designs use feedback to keep a mass from moving — electronics in seismometer stage.

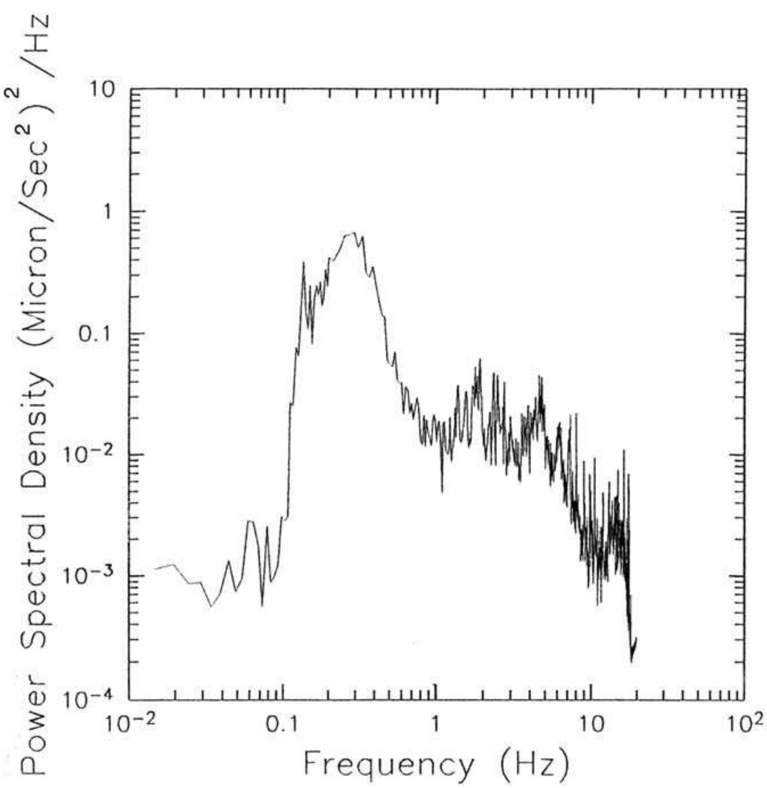
Nonlinearities come from large excursions, heterogeneities of the magnetic field, and "violin modes" (figure below). Electronics also have limited dynamic range. One should also be aware of the fact that calibrations can change with time.



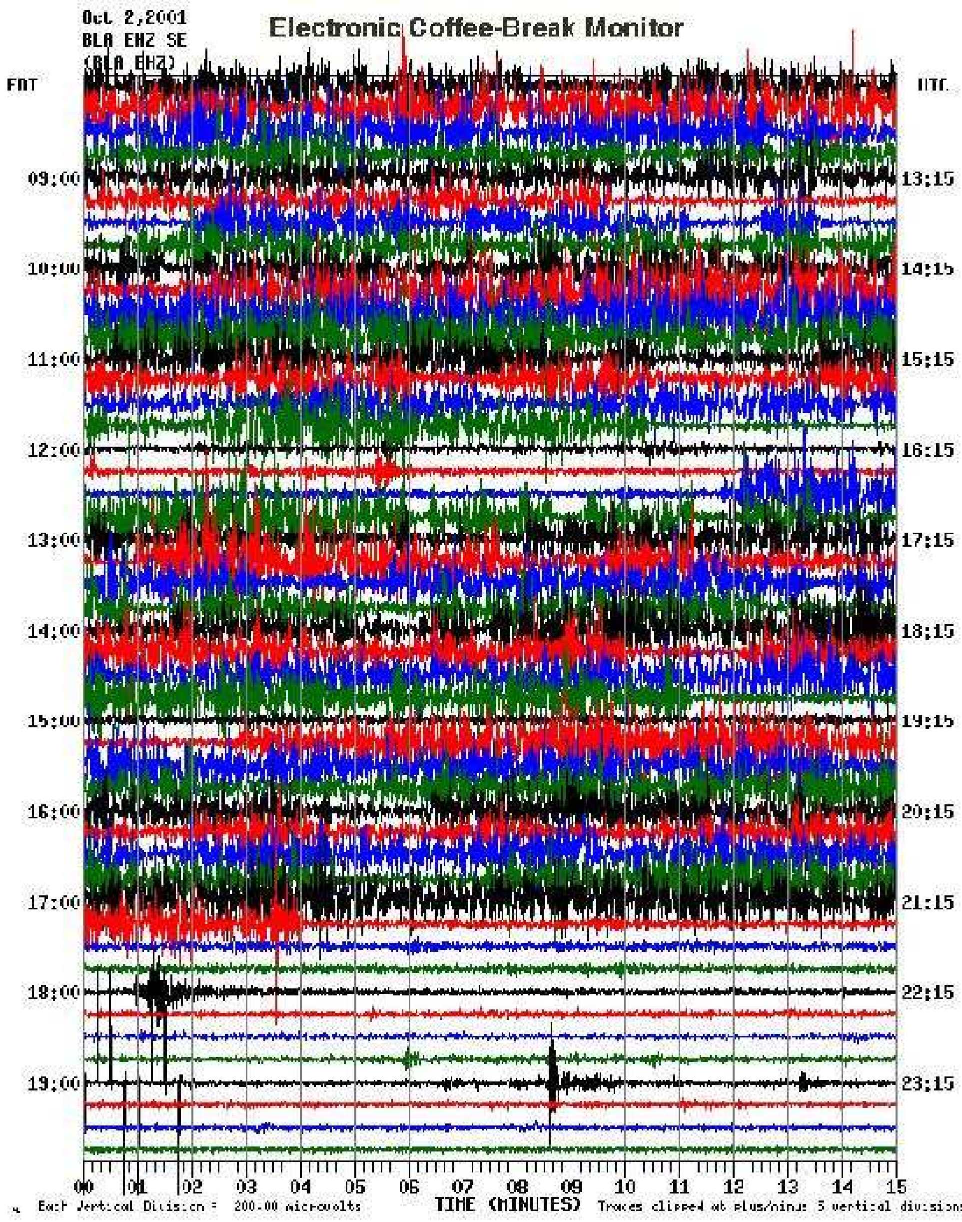
Violin Mode: **(a)** Typical record of a nearby earthquake recorded by a standard long-period seismograph. **(b)** Bow-stringing of a helical spring (Aki & Richards, Figs. 10.21 and 10.22)

Local noise often restricts the potential dynamic range. (See figure on the next page.) Site choice important — away from cultural noise or 300’ down are strategies. Maximum ground noise at around 6 second period 9see below). Must high-pass broadband record to see *P* for small events (see next page).

The figure below shows the noise spectrum (acceleration amplitude squared) from a vertical-component seismograph at BLA on a “typical” winter night (January 2, 1993). The maximum in the noise spectrum comes from microseisms, which have a period of around 6 s. These are caused by sea waves, and tend to be larger in the winter because of winter storms. A plot for a “typical” summer night would be similar, except the microseism noise peak is down by a factor of ten. Individual stations often have very narrow noise spikes caused by local noise sources, such as power lines or resonances caused by the local stratigraphy.

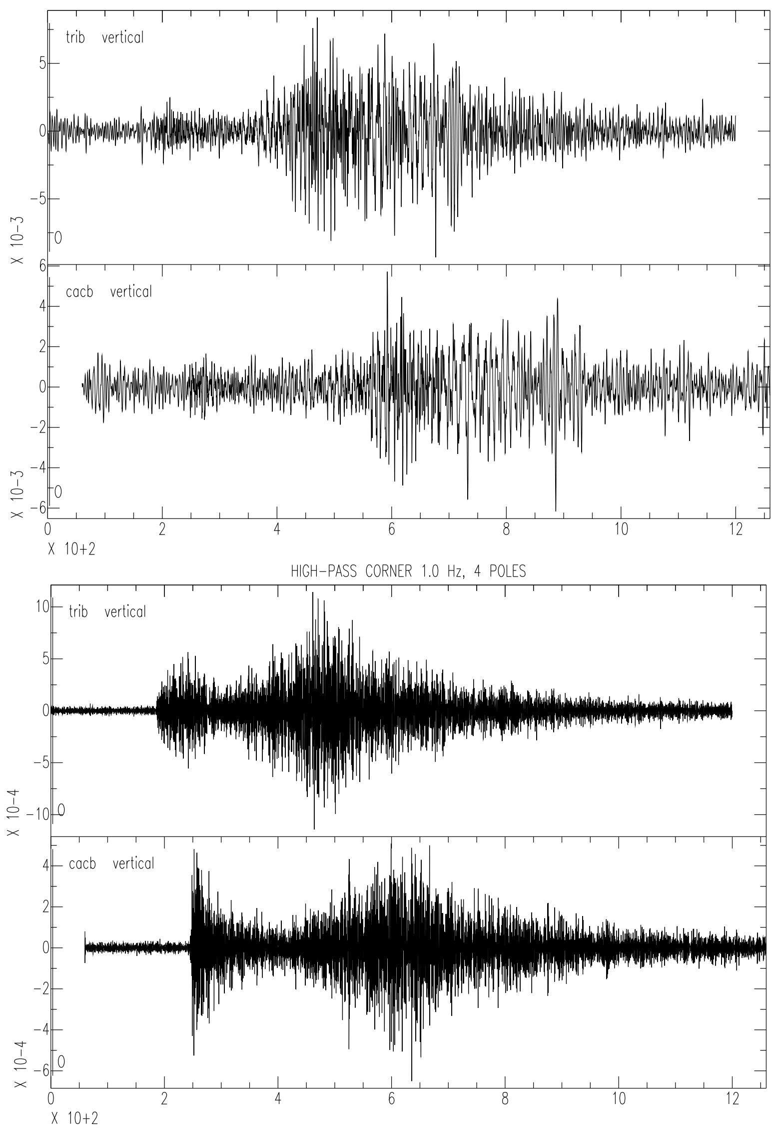


The analog signal is digitized with a sampling rate of hundreds of samples per second. It is then resampled at 40 sps ($\delta t = 0.025$ s). During the resampling, the digital signal is lowpass filtered with a sharp digital filter to protect against aliasing. This filtering — not the ground acceleration — produces the very sharp falloff at the highest frequencies. The Nyquist frequency is 20 Hz (= $1/(2\delta t)$).



Recording by a short-period seismometer at BLA during a time when bulldozers were leveling the ground in a field about a quarter mile away. Note that it is easy to monitor the workers' coffee breaks. During the quiet times, one can see occasional signals which are probably mine blasts. The gain setting for the seismograph is determined by the (normal) local noise.

High-Pass to See the P



Dynamic Range and Gain Ranging

Bits	Ratio	dB
2	$2^2 = 4$	12
4	$2^4 = 16$	24
8	$2^8 = 256$	48
12	$2^{12} = 4,096$	72
16	$2^{16} = 65,536$	96
24	$2^{24} = 16,777,216$	144

$$1 \text{ dB} = 20 \log_{10} \frac{A}{A_{ref}}$$

For a fixed word length, some resolution can be sacrificed by using some bits to change the gain setting. Increases the “effective” dynamic range.

A useful analogy is to think of a number in scientific notation shown in the display of a scientific calculator with space for 10 characters. Say 4 of those characters are used for the exponential — e-07, for example. These are analogous to the gain-ranging bits. The remaining six characters give you your resolution. By using 5 characters for the exponential you could handle a larger range of numbers, but you would sacrifice resolution.

Another analogy is to take a meter stick with markings every 1 mm. If one shrunk the ruler to 10 cm in total length, the total number of markings would be the same (1000), but they would now be every 0.1 mm. The new ruler has better resolution, but cannot handle as large a distance compared with the first one.

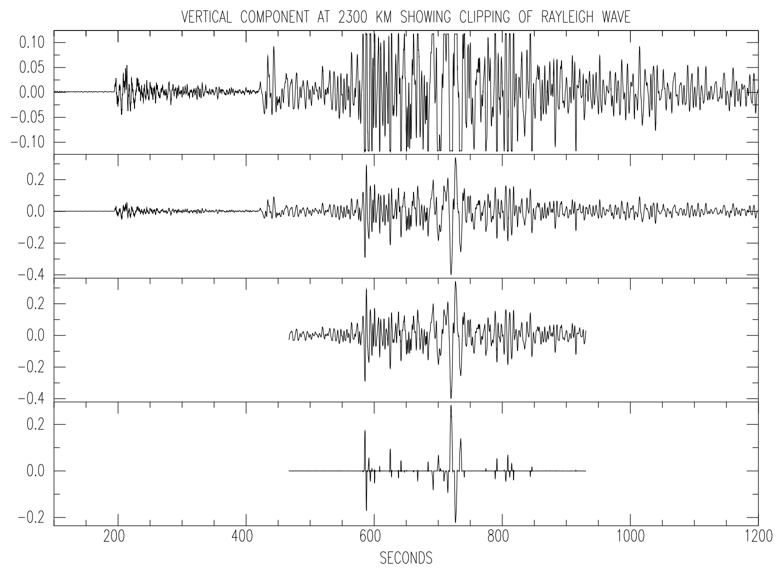
Data with which I have worked from Brazilian stations use a data acquisition system with two 16-bit digitizers per channel, one recording continuously and the other digitizing continuously but recorded only when the signal reaches about 90% of saturation. The second channel is a “low gain” channel with a gain setting a factor of 32 less than the continuously recording channel. The low-gain channel is buffered so that when it is kept, one can have about 60 seconds of data included before the threshold is reached.

If the recording electronics for the two data streams per channel were perfectly matched, the unsaturated parts of the signals recorded in common should be identical. In fact, I find them differing in both gain (magnification) and offset. One explanation for this is that electrical components generally have at best a 1% tolerance.

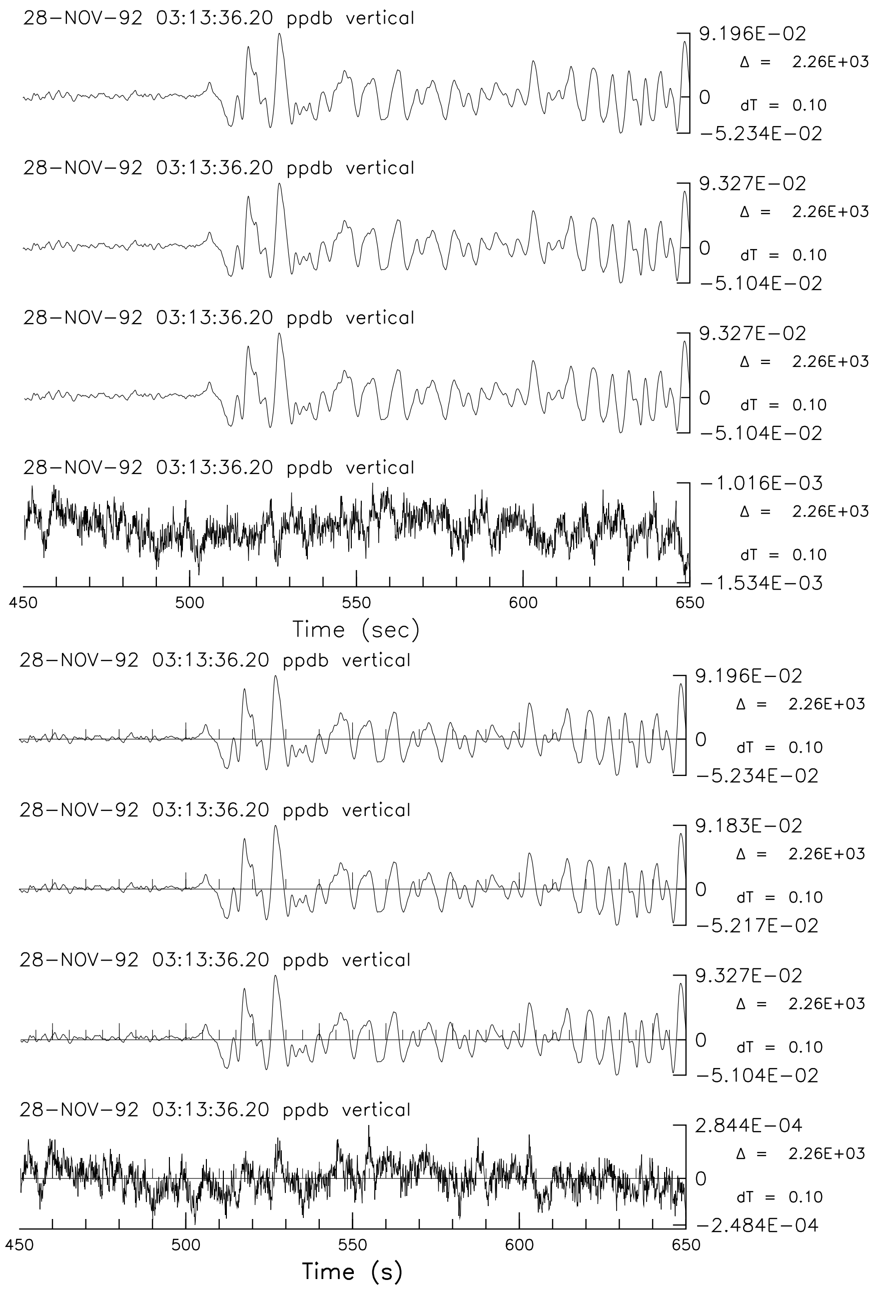
Using a least-squares technique, I correct for the relative gain and offset. The accompanying three figures show the effect. The first shows the saturation for the largest amplitudes. But for surface-wave analysis, I want to start before the low-gain channel starts being recorded. The second set of traces shows that there is a systematic error if one does not correct for gain and offset. The third set is the same as the second except I have applied my correction procedure.

In principle, the maximum amplitude which can be captured without clipping increases by a factor of 32, with a cost that for the time-section replaced with the low-gain channel, the resolution decreases by the same factor. Effectively, this is a 17-bit recording system with 16 bits used for resolution and one bit used for gain ranging. But with a gain-ranging system, there is no way to compare the relative calibrations of the two channels as they never are recording at the same time. Plus, the switching between channels in a gain-ranging system may not work perfectly.

In the example shown below in the figure, the maximum amplitude at which clipping happened is at 0.117 units. The low-gain channel has a maximum about three times as large, so one could have gotten away with a factor of four rather than 32 for this event.



Top trace: high-gain data stream showing clipping. Third trace from top: low-gain data stream showing no saturation but a limited recording time. Bottom trace: Difference between high-gain and low-gain data streams. Second trace from top: Composite data stream formed by replacing the high-gain channel with the low-gain channel. There is an implicit assumption that the two data streams are calibrated — would record the same amplitude and offset in unsaturated parts of the record. The accompanying figures demonstrate that such is not the case for station PPDB.



Top set: No correction. Bottom set: Corrected. Note ranges in amplitude in fourth traces.